Geometry and freeform architecture

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During the last decade, the geometric aspects of freeform architecture have defined a field of applications which is systematically explored and which conversely serves as inspiration for new mathematical research. This paper discusses topics relevant to the realization of freeform skins by various means (flat and curved panels, straight and curved members, masonry, etc.) and illuminates the interrelations of those questions with theory, in particular discrete differential geometry and discrete conformal geometry.

1 Introduction

A substantial part of mathematics is inspired by problems which originate outside the field. In this paper we deal with outside inspiration from a rather unlikely source, namely *architecture*. We are not interested in the more obvious ways mathematics is employed in today's ambitious freeform architecture (see Figure 1) which include finite element analysis and tools for computer-aided design. Rather, our topic is the unexpected interplay of geometry with the spatial decomposition of freeform architecture into beams, panels, bricks and other physical and virtual building blocks. As it turns out, the mathematical questions which arise in this context proved very attractive, and the mundane objects of building construction apparently are connected to several well-developed mathematical theories, in particular discrete differential geometry.

The design dilemma. Architecture as a field of applications has some aspects different from most of applied mathematics. Usually having a unique solution to a problem is considered a satisfactory result. This is not the case here, because architectural design is *art*, and something as deterministic as a unique mathematical solution of a problem eliminates design freedom from the creative process. We are going to illustrate this dilemma by means of a recent project on the Eiffel tower.

The interplay of disciplines. We demonstrate the interaction between mathematics and applications at hand of questions which occur in practice and their answers. We demonstrate how a question Q, phrased in terms of engineering and architecture, is transformed into a specific

Figure 1. Freeform architecture. The Yas Marina Hotel in Abu Dhabi illustrates the decomposition of a smooth skin into straight elements which are arranged in the manner of a torsion-free support structure. The practical implication of this geometric term is easy manufacturing of nodes (image courtesy Waagner-Biro Stahlbau).

mathematical question Q^* which has an answer A^* in mathematical terms. This information is translated back into an answer A to the original question. Simplified examples of this procedure are the following:

Q_1: Can we realize a given freeform skin as a steel-glass construction with straight beams and flat quadrilateral panels?
Q_1^*: Can a given surface Φ be approximated by a discrete- conjugate surface?
A_1^*: Yes, but edges have to follow a conjugate curve network of Φ.
A_1: Yes, but the beams (up to their spacing) are determined by the given skin.

Q_2: For a steel-glass construction with triangular panels, can we move the nodes within the given reference surface, such that angles become $\approx 60°$?
Q_2^*: Is there a conformal triangulation of a surface Φ which is combinatorially equivalent to a given triangulation (V, E, F)?
A_2^*: Yes if the combinatorial conformal class of (V, E, F) matches the geometric conformal class of Φ.
A_2: Yes if the surface does not have topological features like holes or handles.

Overview of the paper. We start in Section 2 with freeform skins with straight members and flat panels, leading to the discrete differential geometry of polyhedral surfaces. Section 3 deals with curved elements, Section 4 with circle patterns and conformal mappings, Section 5 with the statics of masonry shells, and finally Section 6 discusses computational tools.

2 Freeform skins with flat panels and straight beams

Freeform skins realized as steel-glass constructions are usually made with straight members and flat panels because of the high cost of curved elements. Often, the flat panels form a watertight skin. Since three points in space always lie in a common plane, but four generic points do not, it is obviously much easier to use triangular panels instead of quadrilaterals. Despite this

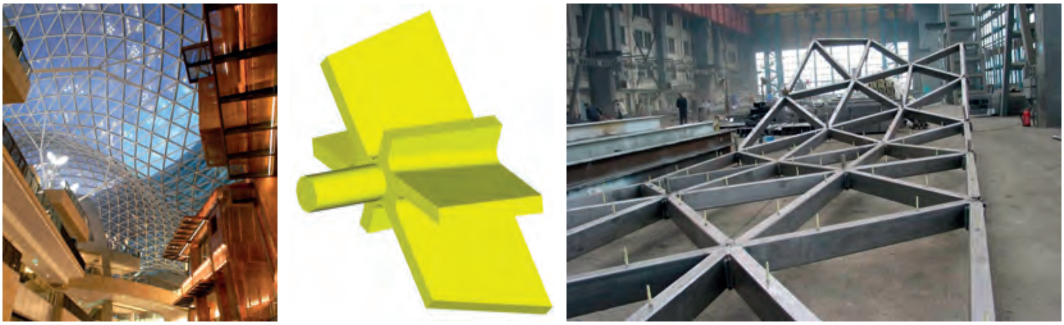

Figure 2. Steel-glass constructions following a triangle mesh can easily model the desired shape of a freeform skin, at the cost of high complexity in the nodes. The *Złote Tarasy* roof in Warszaw (left) is welded from straight pieces and spider-like node connectors which have been plasma-cut from a thick plate (images courtesy Waagner-Biro Stahlbau).

difficulty, the past decade has seen much research in the geometry of freeform skins based on *quadrilateral* panels. This is because they have distinct advantages over triangular ones – fewer members per node, fewer members per unit of surface area, fewer parts and lighter construction (see Figure 2).

2.1 Meshes

We introduce a bit of terminology: A *triangle mesh* is a union of triangles which form a surface, and we imagine that the edges of triangles guide the members of a steel-glass construction. The triangular faces serve as glass panels. Similarly, *quad* meshes are defined, as well as general meshes without any restrictions on the valence of faces. We use the term *planar quad mesh* to emphasize that panels are flat. Dropping the requirement of planarity of faces leads to general meshes whose edges are still straight. We use V for the set of vertices, E for the edges, and F for the faces. The exact definition of "mesh" follows below.

Meshes from the mathematical viewpoint. While a triangle mesh is simply a 2D simplicial complex of manifold topology, a general mesh is defined as follows. This definition is engineered to allow certain degeneracies, e.g. coinciding vertices.

DEFINITION 1. *A mesh in $\mathbb{R}^d$ consists of a two-dimensional polygonal complex (V, E, F) with vertex set V, edge set E, and face set F homeomorphic to a surface with boundary. In addition, each vertex $i \in V$ is assigned a position $v_i \in \mathbb{R}^d$ and each edge $ij \in E$ is assigned a straight line e_{ij} such that $v_i, v_j \in e_{ij}$.*

We say the mesh is a polyhedral surface if it has planar faces, i.e., for each face there is a plane which contains all vertices v_i incident with that face.

2.2 Support structures

An important concept are the so-called torsion-free support structures associated with meshes [30]. Figure 3 shows an example, namely an arrangement of flat quadrilateral panels along the

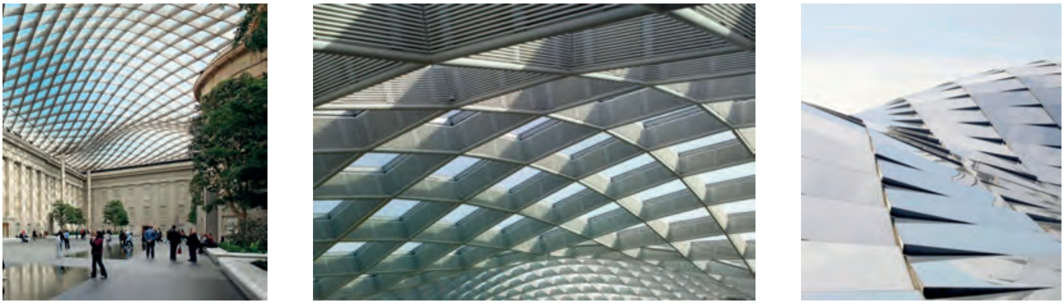

Figure 3. Physical torsion-free support structures. The roof of the Robert and Arlene Kogod Courtyard in the Smithsonian American Art Museum exhibits a mesh with quadrilateral faces and an associated support structure. The faces of the mesh are not planar – only the view from outside reveals that the planar glass panels which function as a roof do not fit together.

edges of a quad mesh (V, E, F) (which does not have planar faces), such that whenever four edges meet in a vertex, the four auxiliary quads meet in a straight line. We define:

DEFINITION 2. *A torsion-free support structure associated with a mesh* (V, E, F) *consists of assignments of a straight line* ℓ_i *to each vertex and a plane* π_{ij} *to each edge, such that* $\ell_i \ni v_i$ *for all vertices* $i \in V$, *and* $\pi_{ij} \supset \ell_i, \ell_j, e_{ij}$ *for all edges* $ij \in E$.

A support structure provides actual support in terms of statics (whence the name), but also has other functions like *shading* [43]. In discrete differential geometry, support structures occur under the name "line congruences".

Benefits of virtual support structures. Figures 1 and 4 illustrate the Yas Marina Hotel in Abu Dhabi, which carries a support structure in a less physical manner: each steel beam has a plane of central symmetry, and for each node these planes intersect in a common node axis, guaranteeing a clean "torsion-free" intersection of beams. This is much better than the complex intersections illustrated by Figure 2.

Combining flat panels and support structures. It would be very desirable from the engineering viewpoint to work with meshes which have both flat faces and torsion-free support structures. They would be able to guide a watertight steel-glass skin and allow for a "torsion-free" intersection of members in nodes such as demonstrated by Figure 4. The following elementary result however says that in order to achieve this, we must essentially do without triangle meshes.

LEMMA 3. *Every mesh can be equipped with trivial support structures where all lines* ℓ_i *and planes* π_{ij} *pass through a fixed point (possibly at infinity).*

Triangle meshes admit only trivial support structures. More precisely this property is enjoyed by every cluster of generic triangular faces which is iteratively grown from a triangular face by adding neighbouring faces which share an edge.

Proof. For an edge ij, there exists the point $x_{ij} = \ell_i \cap \ell_j$ (possibly at infinity), because ℓ_i, ℓ_j lie in the common plane π_{ij}. If ijk is a face, then $x_{ij} = \ell_i \cap \ell_j = (\pi_{ik} \cap \pi_{jk}) \cap (\pi_{ij} \cap \pi_{jk}) = \pi_{ij} \cap \pi_{ik} \cap \pi_{jk}$ implying that $x_{ij} = x_{ik} = x_{jk} \implies$ all axes incident with the face ijk pass through

Figure 4. Torsion-free support structure. For each edge ij and vertex i of a quadrilateral mesh, we have a plane π_{ij} and a line ℓ_i such that e_{ij}, ℓ_i, ℓ_j are contained in π_{ij} (at left, image courtesy Evolute). This support structure guides members and nodes in the outer hull of the Yas Marina hotel in Abu Dhabi, so that members have a nice intersection in each node (at right, image courtesy Waagner-Biro Stahlbau).

a common point. For faces sharing an edge that point obviously is the same, which proves the statement. □

Lemma 3 has far-reaching consequences since it expresses the incompatibility of two very desirable properties of freeform architectural designs. On the one hand frequently a freeform skin is to be watertight, acting as a roof, which for financial reasons imposes the constraint of planar faces. Unfortunately the planarity constraint is difficult to satisfy unless triangular faces are employed. On the other hand, triangle meshes have disadvantages: We already mentioned the large number of members. Lemma 3 states that only in special circumstances it is possible to reduce node complexity by aligning beams with the planes of a support structure.

2.3 Quadrilateral meshes with flat faces

Research related to meshes with planar faces is not new, as proved by the 1970 textbook [33] on *difference geometry* by R. Sauer which in particular summarizes earlier work starting in the 1930's. That work was pioneering for discrete differential geometry, which meanwhile is a highly developed area [11]. Relevant to the present survey, questions concerning quad meshes with planar faces ten years ago marked the starting point of a line of research motivated by problems in engineering and architecture [26], which again led to new developments in discrete differential geometry, see Section 2.4 below.

The meaning of "freeform". Research on quad meshes with flat faces has been rewarding mathematically, but unfortunately hardly any truly freeform meshes of that type have been realized as buildings. Their welcome qualities have nevertheless been used for impressive architecture, but meshes built so far enjoy special geometric properties (like rotational symmetry) which allows us to describe their shape using much less information than would be required in general, see Figure 5.

Smoothness limiting design freedom. A typical situation in the design process of freeform architecture is the following: A certain mesh has been created whose visual appearance fits the

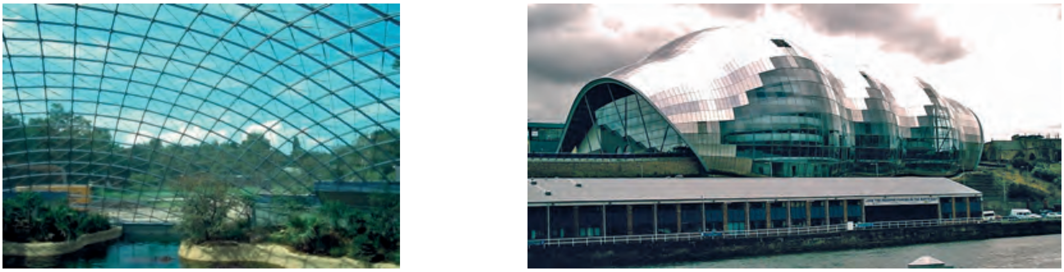

Figure 5. Not entirely freeform surfaces. Left: The hippo house in the Berlin zoo is based on a quadrilateral mesh with flat faces, but is not freeform in the strict sense. Its faces are parallelograms, so the mesh is generated by parallel translation of one polyline along another one. Mathematically, vertices $v_{i,j}$ have the form $v_{i,j} = a_i + b_j$.
Right: the Sage Gateshead building on the river Tyne, UK, is based on a sequence of polylines which are scaled images of each other, similar to a mesh with rotational symmetry.

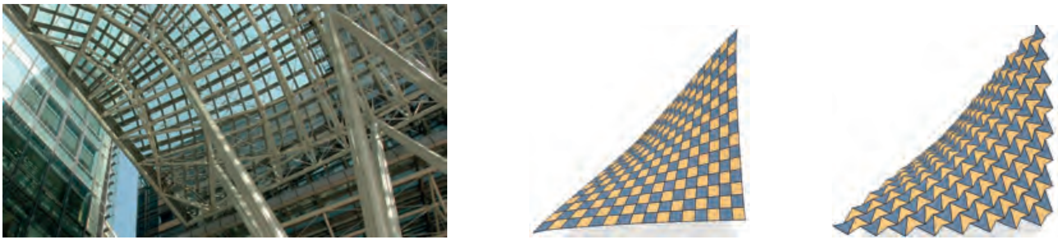

Figure 6. Quad meshes. Left: The canopy at “Tokyo Midtown” is based on a quad mesh with planar faces. *Center*: This quad mesh has nonplanar faces, and all meshes nearby which have planar faces are far from smooth. *Right:* This mesh has planar faces and is no direct discrete analogue of a continuous smooth surface parametrization (for such regular patterns see [24]).

intentions of the designer and which eventually is to be realized as a steel-glass construction with flat panels. The designer therefore wants the vertex positions to be altered a little bit so that the faces of the mesh become planar, but its visual appearance does not change. Unfortunately this problem is typically not solvable. This is not because the nonlinear nature of this problem prevents a numerical solution – the reasons are deeper and of a more fundamental, geometric nature: For example, it is known that a “smooth” mesh of regular quad combinatorics which follows a smooth surface parametrization can have planar faces only if its edges are aligned with a so-called conjugate curve network of the reference surface. The network of principal curvature lines is the major example of that, cf. Figure 8. Since its singularities are shared (in a way) by all conjugate curve networks [45], the principal curvature lines already give a good impression of what a planar quad mesh approximating a given surface must look like. If the designer’s mesh is not conjugate, there is no smooth mesh nearby which has planar faces – see Figure 6. There is no easy way out of this dilemma other than reverting to triangular faces, or redesigning the mesh entirely so that its edges follow a conjugate curve network, or forgoing smoothness.