# Some geometries to describe nature

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*Since ancient times, the development of mathematics has been inspired, at least in part, by the need to provide models in other sciences, and that of describing and understanding the world around us. In this note, we concentrate on the shapes of nature and introduce two related geometries that play an important role in contemporary science. Fractal geometry allows describing a wide range of shapes in nature. In 1973, Harry Blum introduced a new geometry well suited to describe animal morphology.*

The range of shapes and patterns in nature is immense. At the larger scale, we can think of the spiral shapes of galaxies, which resemble those of some hurricanes. There are many other shapes on Earth, from mountains to dunes, to oceans, to clouds, and to networks of rivers. Coming to still smaller scales, we observe the many shapes of life, be it vegetation or animals. At still smaller scales, we explore the complexity of the capillary network and bronchial tubes in the lungs. We discover common features: for instance, the capillary network resembles a river network. Can we measure these resemblances? Can we explain how these complicated shapes are produced and why some similarities of shapes occur in many different situations?

Traditional geometry uses lines, circles, smooth curves and surfaces to describe nature. This is sufficient to describe remarkable shapes like the logarithmic spiral of a nautilus, or the spiral shapes of hurricanes and galaxies, and to explain why they occur. If we want to describe the DNA strands, then the help of knot theory is welcome since the DNA strands are so long that they require entanglement to be packed in the cell. The traditional geometry is not sufficient to describe the shapes of many plants like trees, ferns, the Romanesco cauliflower, or the branch of parsley. We need a geometry to describe the plants. This geometry is the *fractal geometry*, created by the visionary mathematician, Benoît Mandelbrot (see for instance [5]).

## 1 Fractal geometry

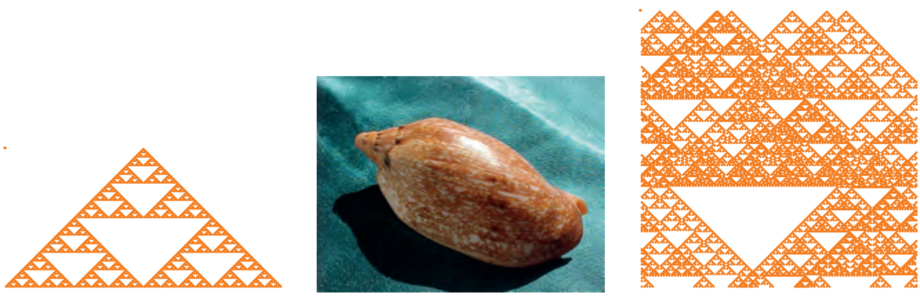
The fern, the Romanesco califlower, or the branch of parsley are very complicated objects. Well, they look very complicated if we observe them through our eyes … Hence, let us put on mathematical glasses and discover their hidden simplicity: they are *self-similar* (Figure 1). This means that if we zoom to some part of an object, then we see the same pattern reappearing at different scales. In practice, this self-similarity means that the object looks complicated just

a The fern b A branch of parsley

*Figure 1.* A fern and a branch of parsley

because we do not look at it from the right point of view. And indeed, self-similar objects can be reconstructed with extremely simple programs if we make use of their self-similar structure (see, for instance, [6] or [8]).

Mathematicians like to focus on simpler models. Hence, we will concentrate on the Sierpiński carpet (see Figure 2a) which is obtained from a plain triangle, deleting the middle triangle, and iterating. We already face a surprise. The Sierpiński carpet is really an object created from the imagination of mathematicians. Well, nature seemed to have borrowed this pattern on Cymbiola Innexa REEVE (see Figure 2b)! And if you look at the computer generated pattern in Figure 2c, it is very similar to the pattern on the shell! We will come back to this pattern later. When introducing fractal geometry to describe the shape of plants, we have done much better. This new geometry can also be used to describe many more shapes of nature including rocky



a Sierpiński carpet b Cymbiola Innexa REEVE c computer generated pattern

*Figure 2.* The Sierpiński carpet, the shell Cymbiola Innexa REEVE (Photo credit: Ian Holden, Schooner Specimen Shells), and a computer generated pattern

coasts, the river networks or the patterns drawn by water flowing on the sand, the deltas of large rivers like Niger, clusters of galaxies or the Milky Way, clouds, frost, the bronchial tubes or the network of capillaries, etc.

We want to understand the geometry of these objects and what characterizes them.

### 1.1 Measuring a fractal

How can we measure the length of a rocky coast? This seems a difficult operation. Indeed, each time we zoom, we see new details appearing, thus adding to the length. Hence, when do we stop zooming?

Again, as mathematicians, we will first study a simple model. This model is the von Koch snowflake (see Figure 3a). Is is obtained by *iteration*, i.e., by repeating the same step(s) again and again. At each step of the iteration, we replace each segment by a group of 4 segments with length equal to $1/3$ of the length of the original segment. If $L$ is the length of the original triangle in Figure 3b, then $\frac{4}{3}L$ is the length of the star of Figure 3c, $\left(\frac{4}{3}\right)^2 L$ is the length of the object in Figure 3d, etc. In particular, this means that at each step the length is multiplied by $\frac{4}{3}$. Since there are an infinite number of steps in the construction, then the length of the von Koch snowflake is infinite even though the curve fits into a finite region of the plane!

As an exercise, you could play the same kind of game and check that the area of the Sierpiński carpet is zero.

We have seen that a curve can have an infinite length even when it lies in a finite region of the plane. In the same way, there exists fractal surfaces with infinitely many peaks that have infinite area and fit in a finite volume. Such properties are useful for nature. For instance, while the outer surface of the small intestine has an approximate area of $0.5\,\mathrm{m}^2$, the inner surface has an approximate area of $300\,\mathrm{m}^2$. The fractal nature of the inner surface allows its large area and favors the intestinal absorption.

The examples above show that length and area give little information on a fractal. The concept which will allow to give more relevant information is that of *dimension*.

### 1.2 Dimension of a fractal object

How do we mathematically define *dimension*? We have an intuitive idea of dimension. Indeed, smooth curves are of dimension 1, smooth surfaces, of dimension 2, and filled volumes, of

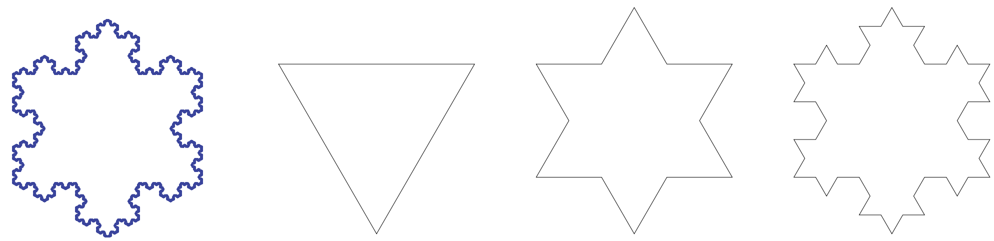



*Figure 3.* Von Koch snowflake and the iteration process to construct it

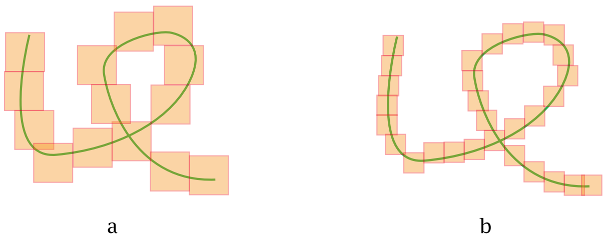



*Figure 4.* Calculating the dimension of a curve using squares of different sizes

dimension 3. Hence, we should give a mathematical definition of dimension, which yields 1 for smooth curves, 2, for smooth surfaces, and 3, for filled volumes. In the context of this paper, we will limit ourselves to dimensions 1 and 2. We want to cover an object in the plane with small squares. (If we would want to define dimension 3 we would use small cubes, but we could have used small cubes for curves and surfaces without changing the dimension!)

*Case of a smooth curve.* (see Figure 4)

- If we take squares with side of half size, then we approximately double the number of squares needed to cover the object.
- If we take squares with side one third of the size, then we approximately triple the number of squares needed to cover the object.
- ...
- If we take squares with side $n$ times smaller, then we approximately need $n$ times more squares to cover the object.

*Case of a surface.* (see Figure 5)

- If we take squares with side of half size, then we approximately need four times more squares to cover the object.
- If we take squares with side one third of the size, then we approximately need nine times more squares to cover the object.

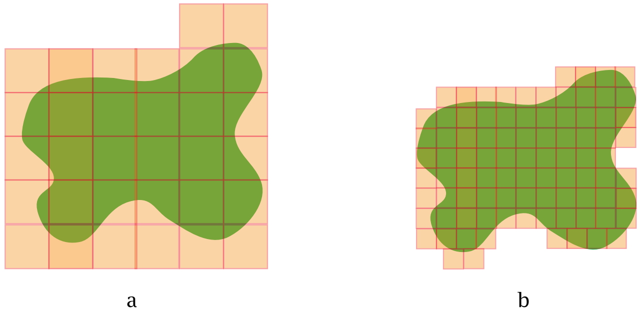



*Figure 5.* Calculating the dimension of a surface using squares of different sizes

- ...
- If we take squares with side $n$ times smaller, then we approximately need $n^2$ times more squares to cover the object.

We can now give the (intuitive) definition of dimension.

DEFINITION. *An object has* dimension $d$ *if, when we take squares (or cubes) with edge $n$ times smaller to cover it, then we need approximately $n^d$ more squares (or cubes) to cover it.*

Not all objects have a dimension. But, the self-similar objects have a dimension which, most often, is not an integer. We assert without proof that the fractal dimension of the von Koch snowflake is $\frac{\ln 4}{\ln 3} \sim 1.26$. Let us now calculate the dimension of the Sierpiński carpet (see Figure 6).

- Let us take a square with side equal to the length of the base (see Figure 6b). It covers the Sierpiński carpet of Figure 6a.
- If we take squares with side of half size, then we need three squares to cover the Sierpiński carpet. Note that $3 = 2^{\frac{\ln 3}{\ln 2}}$ (Figure 6c).
- if we take squares with side one fourth of the size, then we need nine squares to cover the Sierpiński carpet. Note that $9 = 4^{\frac{\ln 3}{\ln 2}}$ (Figure 6d).
- If we take squares with side one eight of the size, then we need 27 squares to cover the Sierpiński carpet. Note that $27 = 8^{\frac{\ln 3}{\ln 2}}$ (Figure 6e).

Hence, it is easy to conclude that the dimension of the Sierpiński carpet of Figure 6a is $d = \frac{\ln 3}{\ln 2} \sim 1.585$. We assert without proof that it is also the dimension of the Sierpiński carpet of Figure 2, even if that is not as obvious (see for instance [7]).

The dimension gives a "measure" of the complexity or density of a fractal. We feel that the Sierpiński carpet is denser than the von Koch snowflake, which looks more like a thickened

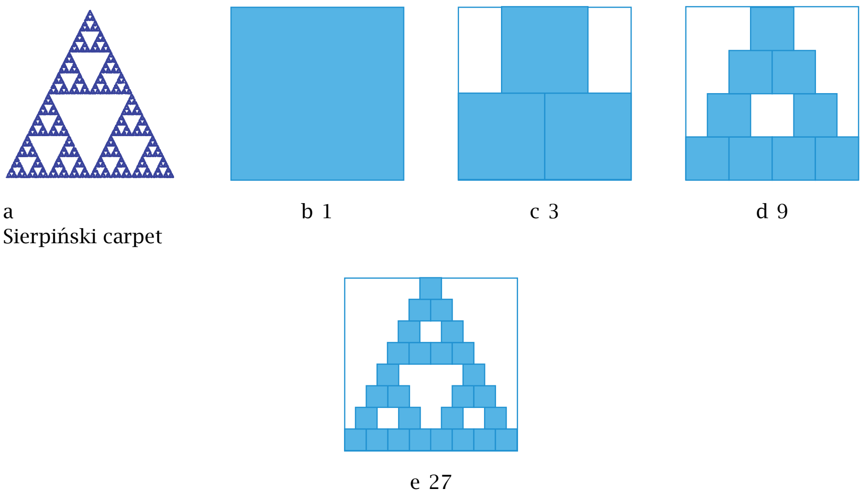



*Figure 6.* Number of squares to cover to cover the Sierpiński carpet appearing in a

curve. This is reflected by the fact that $\frac{\ln 3}{\ln 2} > \frac{\ln 4}{\ln 3}$, i.e., the dimension of the Sierpiński carpet is larger than the dimension of the von Koch snowflake.

### 1.3 Applications

There are many applications in a wide range of disciplines. Let us mention two in medicine.

The capillary network is not the same in the neighborhood of a tumor as elsewhere in the body. Research is carried on this, and in particular on its fractal dimension, in order to improve diagnosis from medical imaging.

High level athletes are more likely to suffer from asthma than the general population. The paper [3] studies the "optimal lung". There are 17 level of bronchial tubes before arriving to the terminal bronchioles followed by the acini involved in air exchange. If the bronchial tubes are too thin, then the pressure increases too much when the air penetrates in the next level of bronchial tubes. But if they are too wide, so that the volume never decreases from one level to the next, then the volume becomes too large. (It would become infinite if we had an infinite number of levels). Hence, the "optimal lung" would have the minimum volume without increasing the pressure. But the rate of variation of pressure close to the optimal lung is quite high, which means that a slight decrease in the diameter of the bronchial tubes induces a substantial increase of the pressure. To overcome this, the human lungs have a higher volume than the theoretical optimal lungs. This buffer provides a safety factor to protect in case of a pathology decreasing the diameter of the bronchial tubes, like asthma. Athletes have lungs generally closer to the theoretical optimal lungs, and hence, are more vulnerable.

### 1.4 Coming back to Cymbiola Innexa REEVE

How do we generate the pattern of Figure 2(c)? The scientists work on the hypothesis that a model of reaction-diffusion generates the pattern. The shell grows from its lower boundary, so the pattern should expand on the bottom side. We have two chemical reactants:

- one activator (colored),
- one inhibitor (uncolored).

The two reactants diffuse. The pattern is sensitive to initial conditions. We can obtain many patterns by varying the initial conditions. The diffusion starts from the top and goes to the bottom, when a new stripe is added on the side of the shell.

Let us present a simplification of the model. We approximate each layer by small squares (pixels). How is determined the color of a pixel? It depends on the color of the two pixels on the level just above, which are adjacent to it by the corner. If the adjacent two pixels are of the same type (both activators or inhibitors), then this generates a pixel with activator (colored). If the adjacent two pixels are of different types (one activator and one inhibitor), then this generates a pixel with inhibitor (uncolored).

This is how we write it mathematically. Activator (or color) is denoted by 1 and inhibitor by 0. The pattern is created following the three rules

$$\begin{cases} 0+0=0, \\ 1+0=0+1=1, \\ 1+1=0, \end{cases} \tag{1}$$